Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 5,6

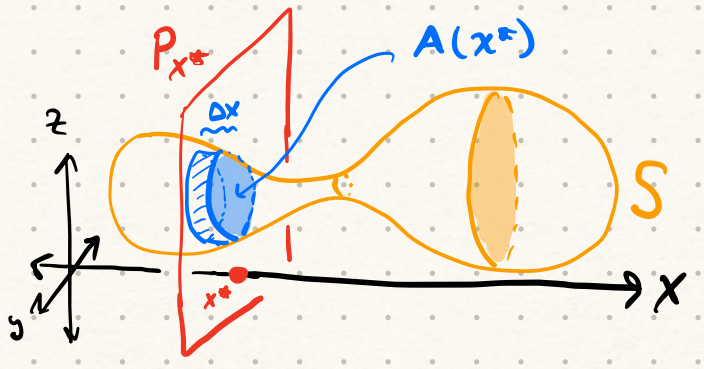
Lecturer: Hunter Stufflebeam

Textbook Sections: 6.2, 6.3

Today's Topics: Solids of Revolution

Recall: In Lecture 4, we introduced a method for calculating the volume of a solid region S by slicing it up and integrating the area of the slices:

$$(\text{volume of } S] := \lim_{N\to\infty} \sum_{i=1}^{N} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx .$$



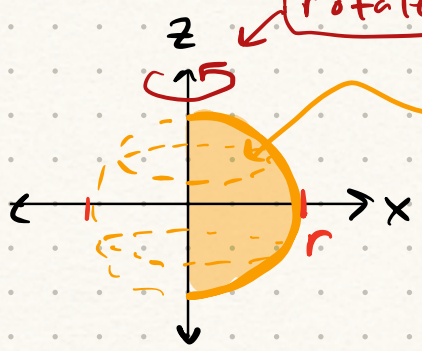
In today's lecture, we will apply this method to a special variety of regions called **solids of revolution**, as well as give a similar, alternative approach to handle some trickier problems of the same sort.

First:

Defintion: A **solid of revolution** is a solid 3D region obtained by revolving a region in a 2D-plane around an axis.

Examples

① Balls

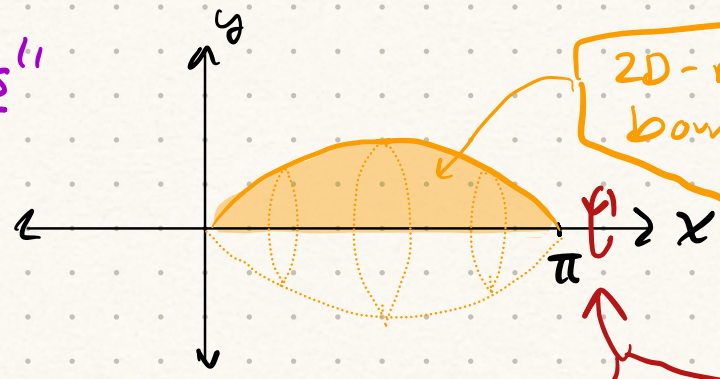


rotate around the z-axis

2D-region in xz plane bounded by the z-axis and the graph of $f(z) = \sqrt{r^2 - z^2}$ from $x = -1$ to 1

We calculated the volume of this region in the last lecture!

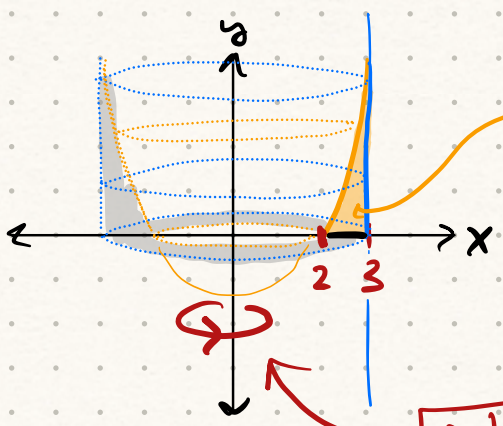
② "Footballs"



2D-region in xy plane bounded by the x-axis and the graph of $f(x) = a \sin x$ from $x = 0$ to π.

rotate around x-axis

③ "Regions with Holes" (Don't erase this for later use)

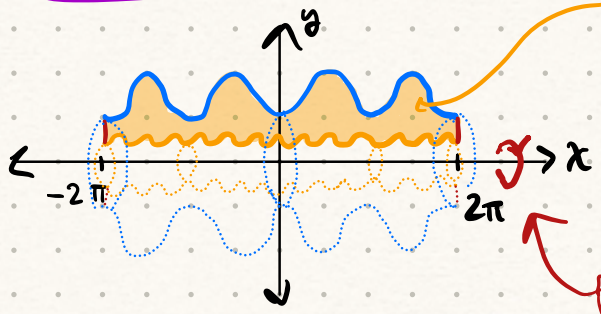


2D region in the xy plane bounded by

$\begin{cases} y = x^2 - 4 \\ x = 2 \text{ and } x = 3 \\ x\text{-axis} \end{cases}$

rotate around the y-axis

④ A candle stick?



2D region in the xy plane bounded by

$\begin{cases} y = \frac{1}{4}\cos\left(\frac{3}{2}x\right) + \frac{1}{2} \\ y = 2 - \cos(x) \\ x = -2\pi \quad x = 2\pi \end{cases}$

rotate around the x-axis

Computations:

(1) We did this last class – notice that if we slice in the direction of the z-axis, the cross section thru z has

$$A(z) = \pi(\text{radius at height } z)^2$$
$$= \pi(r^2 - z^2)$$

So that $$\text{volume } S = \int_{-r}^{r} A(z)\,dz = \int_{-r}^{r} \pi(r^2 - z^2)\,dz = \frac{4}{3}\pi r^3.$$

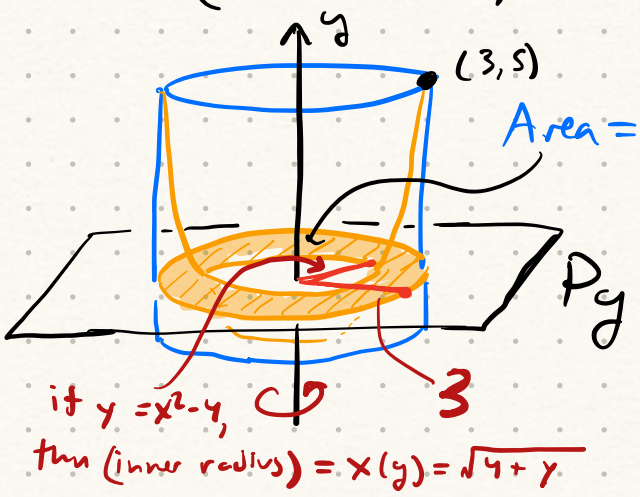
(2) Here we use a different axis!

$$\text{volum } S = \int_0^{\pi} A(x)\,dx = \int_0^{\pi} \pi(a\sin x)^2\,dx = \pi a^2 \int_0^{\pi} \sin^2 x\,dx$$

$$= \pi a^2 \int_0^{\pi} \tfrac{1}{2}(1 - \cos 2x)\,dx = \pi a^2 \cdot \left(\tfrac{x}{2} - \tfrac{1}{4}\sin 2x\right)\Big|_0^{\pi}$$
$$= \frac{(\pi a)^2}{2}$$

$\left(\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha\right.$
$\left.\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\right)$

(3) In this case, our cross-sections have holes! (Sometimes, this is called the "washer method:"



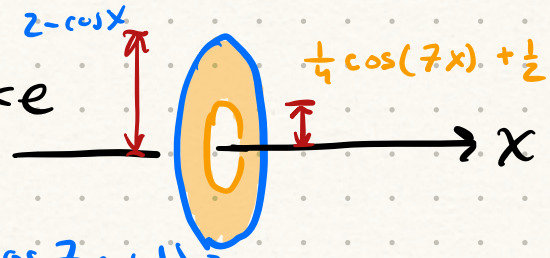
Area $= A(y) =$ (area of washer)
$=$ (area of outer disk) $-$ (area of inner disk)
$= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$
$= \pi(3)^2 - \pi(\sqrt{4+y})^2$
$= \pi(9 - 4 - y)$
$= \pi(5 - y)$

⚠ We also need bounds on y: we start at $y = 0$ (the x-axis) and integrate up to where the boundary curves $y = x^2 - 4$ and $x = 3$ meet (when $(x, y) = (3, 5)$).

Thus:

$$\text{volume } S = \int_0^5 A(y)\,dy = \int_0^5 \pi(5 - y)\,dy = \frac{25\pi}{2}.$$

(4) Here, our washers look like



so $A(x) = \pi(2-\cos x)^2 - \pi\left(\frac{1}{4}\cos 7x + \frac{1}{2}\right)^2$

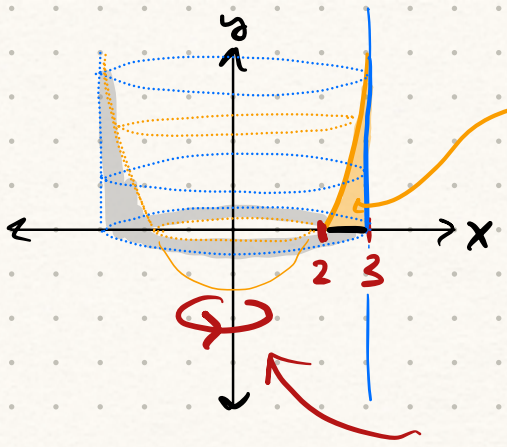
$= \cdots$

Key Point: The main challenge in doing these problems is finding the right setup — after that, the problem is "just calculus."

So far we have seen one way to compute volumes of regions "by slicing them up." However, for some problems there is another approach which is better suited!

The "Shell" Method:

Let's revisit example (3)



2D region in the xy plane bounded by

$\begin{cases} y = x^2 - 4 \\ x = 2 \text{ and } x = 3 \\ x\text{-axis} \end{cases}$

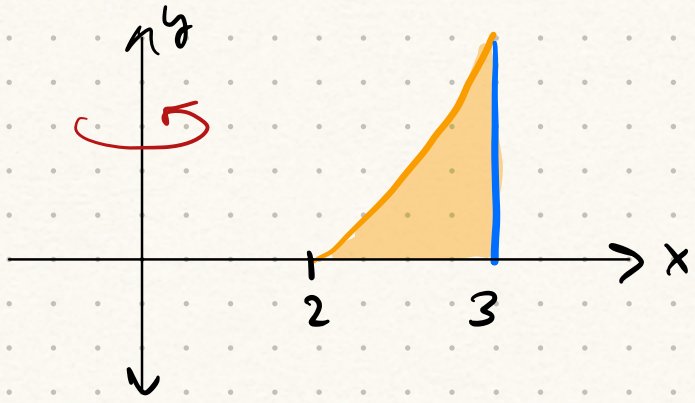
We computed the volume by slicing orthogonally to the y-axis and integrating the areas of these washers, and in doing so we had to invert the function $y = x^2 - 4$ to find the inner radius of the washers.

This function wasn't too hard to invert, but imagine if the function was a lot more complicated.

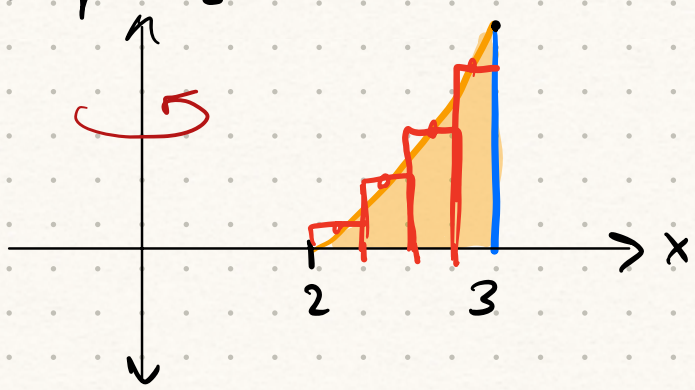
In such a case, it might be easier to use a different approach called the shell-method:

In this method, we take the following POV for approximating the volume of a solid of revolution:

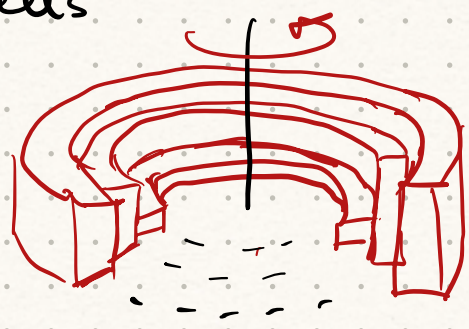
- The solid S is obtained by rotating a region around an axis: Here, we notice that this region is the area between the x-axis and the graph of $y = x^2 - 4$ over the interval $[2,3]$.



- Subdivide $[2,3]$ into subintervals, and consider the Riemann sum approximation by rectangles over these subintervals (suppose we use midpoints for test points:



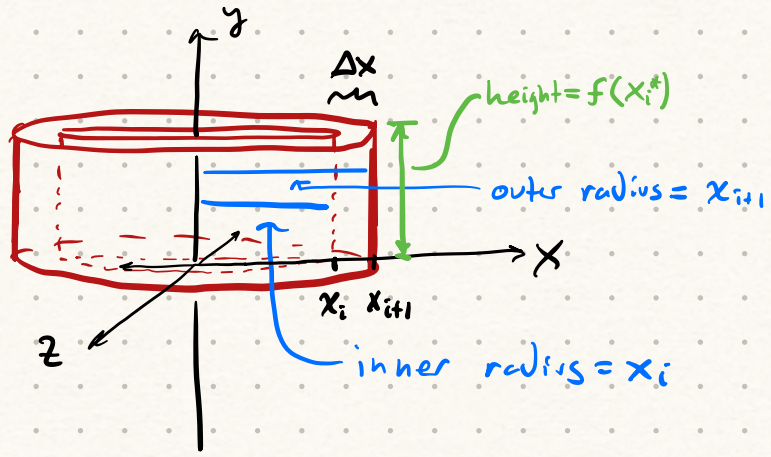
- If we rotate these approximating rectangles around the axis, we obtain an approximation to the solid of revolution by "an onion of cylindrical shells"



- The point: We can calculate the volume of each of these shells without much difficulty. Consider one of them

midpoint rule:

$$x_i^* = \frac{x_i + x_{i+1}}{2}$$



volume of shell = (volume of outer cylinder) − (volume of inner cylinder)

$$= \pi(\text{outer radius})^2(\text{height}) - \pi(\text{inner radius})^2(\text{height})$$

$$= \pi(x_{i+1})^2 f(x_i^*) - \pi(x_i)^2 f(x_i^*)$$

$$= \pi f(x_i^*)\left(x_{i+1}^2 - x_i^2\right)$$

$$= \pi f(x_i^*)(x_{i+1} + x_i)(x_{i+1} - x_i)$$

$$= 2\pi f(x_i^*)\cdot x_i^* \Delta x$$

So

$$\text{volume of } S \approx \sum_{i=1}^{N} (\text{volume of shell } i)$$

$$= \sum_{i=1}^{N} 2\pi x_i^* f(x_i^*) \Delta x$$

and as $N \to \infty$, we get a new formula

$$\text{volume } S = \int_a^b 2\pi x f(x)\, dx$$

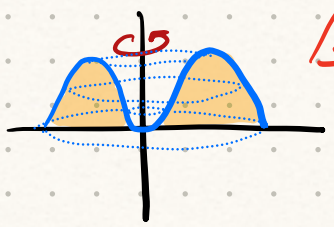
Example (3) revisited

We have

$$\text{volume } S = \int_2^3 2\pi x f(x)\, dx = \int_2^3 2\pi x (x^2 - 4)\, dx = \frac{25\pi}{2}.$$

A new example:

Let S be the region bounded by $y = 2x^2 - x^3$ and $x = 0$, rotated around the y-axis:

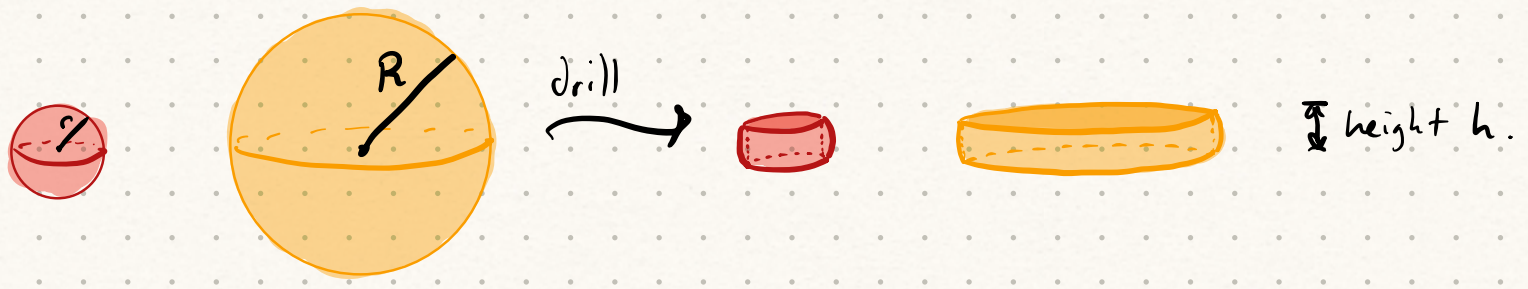


⚠ It would be pretty messy to do this with washers, but...

$$\text{vol } S = \int_0^2 2\pi x (2x^2 - x^3)\, dx = \cdots = \frac{16\pi}{5}.$$

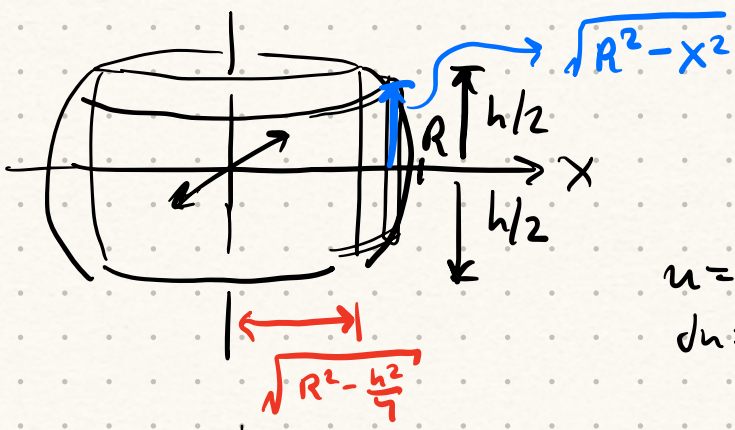
Bonus Topic: The Napkin Ring Problem

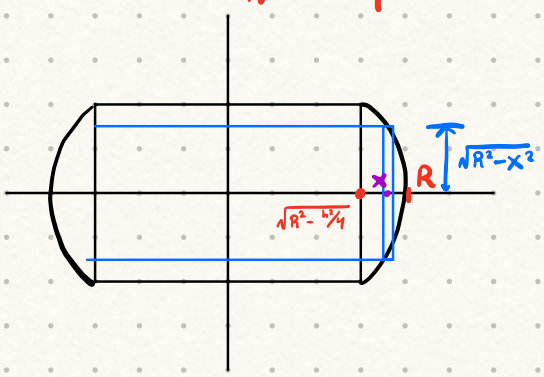
Setup:
- Suppose we have two balls of different diameters (for example, a baseball and the sun).
- Drill holes straight thru each ball (of different sizes) along a diameter, so that the remaining "napkin rings" have the same height:



Question: Which ring has bigger volume?
Give a formula in terms of R and h.

Soln: use cylindrical shells:





$$\text{volume} = \int_{\sqrt{R^2-h^2/4}}^{R} 4\pi x\sqrt{R^2-x^2}\,dx$$

$u = R^2 - x^2$
$du = -2x\,dx$

$$= -2\pi \int_{h^2/4}^{0} \sqrt{u}\,du$$

$$= 2\pi \int_{0}^{h^2/4} \sqrt{u}\,du$$

$$= \frac{\pi h^3}{6}$$

⚠ The volume DOESN'T depend on R!

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